

# Quadratic $\mathcal{PT}$ -symmetric operators with real spectrum and similarity to self-adjoint operators

Emanuela Caliceti

Dipartimento di Matematica  
Università di Bologna  
40127 Bologna, Italy  
Emanuela.Caliceti@unibo.it

Sandro Graffi

Dipartimento di Matematica  
Università di Bologna  
40127 Bologna, Italy  
Sandro.Graffi@unibo.it

Michael Hitrik

Department of Mathematics  
University of California  
Los Angeles  
CA 90095-1555, USA  
hitrik@math.ucla.edu

Johannes Sjöstrand

IMB, Université de Bourgogne  
9, Av. A. Savary, BP 47870  
FR-21078 Dijon, France  
and UMR 5584 CNRS  
johannes.sjostrand@u-bourgogne.fr

**Abstract:** It is established that a  $\mathcal{PT}$ -symmetric elliptic quadratic differential operator with real spectrum is similar to a self-adjoint operator precisely when the associated fundamental matrix has no Jordan blocks.

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## 1 Introduction and statement of result

$\mathcal{PT}$ -symmetric operators in quantum mechanics (for general references on  $\mathcal{PT}$ -symmetry and quantum mechanics, see e.g. [5]) are those operators on  $\mathbf{R}^n$  left invariant by a successive application of the parity operator  $\mathcal{P}$ , acting on wave functions as

$$(\mathcal{P}\psi)(x) = \psi((-1)^{j_1}x_1, \dots, (-1)^{j_n}x_n), \quad j_k = 0, 1, \text{ not all equal to } 0,$$

and of the time-reversal symmetry, acting as  $(\mathcal{T}\psi)(x) = \overline{\psi(x)}$ . The  $\mathcal{PT}$ -symmetry of an operator is called *exact* if its spectrum is purely real, see e.g. [4]. Generally speaking, the reality of the spectrum, and thus the exact  $\mathcal{PT}$ -symmetry, cannot follow by a unitary equivalence to a self-adjoint operator, since a  $\mathcal{PT}$ -symmetric operator is not necessarily self-adjoint, and actually, the standard examples are not self-adjoint and not even normal operators. However, it could follow by a *similarity* to a self-adjoint operator. Let us therefore proceed now to discuss the notion of similarity for two unbounded linear operators. For the limited purposes of this paper, here we shall only consider a rather particular abstract situation, introducing assumptions which will be satisfied in the specific instances below.

Let  $\mathcal{H}_j$ ,  $j = 1, 2$ , be complex separable Hilbert spaces, and let  $\mathcal{A}_j : \mathcal{H}_j \rightarrow \mathcal{H}_j$  be closed densely defined operators such that for  $j = 1, 2$ , the spectrum  $\text{Spec}(\mathcal{A}_j) \subset \mathbf{C}$  is discrete, consisting of eigenvalues of finite algebraic multiplicity. When  $\lambda \in \text{Spec}(\mathcal{A}_j)$ , we let  $E_\lambda^{(j)} \subset \mathcal{D}(\mathcal{A}_j)$  be the finite-dimensional space of generalized eigenvectors, corresponding to  $\mathcal{A}_j$ ,  $\lambda$ , so that

$$E_\lambda^{(j)} = \text{Ker} (\mathcal{A}_j - \lambda)^N,$$

if  $N \geq N(\lambda, j)$  is large enough.

Let  $\mathcal{S}_j \subset \mathcal{D}(\mathcal{A}_j)$ ,  $j = 1, 2$ , be linear subspaces such that  $\mathcal{A}_j(\mathcal{S}_j) \subset \mathcal{S}_j$ , and  $E_\lambda^{(j)} \subset \mathcal{S}_j$ , for each  $\lambda \in \text{Spec}(\mathcal{A}_j)$ . Assume that  $S : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  is a linear bijection such that

$$S\mathcal{A}_1 = \mathcal{A}_2S \quad \text{on } \mathcal{S}_1, \quad (1.1)$$

or equivalently, such that

$$\mathcal{A}_1S^{-1} = S^{-1}\mathcal{A}_2 \quad \text{on } \mathcal{S}_2. \quad (1.2)$$

We shall then say that the operators  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are related by the similarity transformation  $S$ , or are *similar*.

**Proposition 1.1** *The similar operators  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are isospectral.*

*Proof:* Let  $u \in E_\lambda^{(1)}$ , so that  $(\mathcal{A}_1 - \lambda)^N u = 0$ , for some  $N \in \mathbf{N}$ . It follows that  $u, (\mathcal{A}_1 - \lambda)u, \dots \in \mathcal{S}_1$ , and we can apply (1.1) to see that  $(\mathcal{A}_2 - \lambda)^N Su = 0$ , so  $Su \in E_\lambda^{(2)}$ . Thus we have shown that  $S(E_\lambda^{(1)}) \subset E_\lambda^{(2)}$ . The same argument applied to  $S^{-1}$  gives  $E_\lambda^{(1)} \supset S^{-1}(E_\lambda^{(2)})$ , and we get  $\text{Spec}(\mathcal{A}_1) = \text{Spec}(\mathcal{A}_2)$ , and  $S(E_\lambda^{(1)}) = E_\lambda^{(2)}$ , for all  $\lambda \in \text{Spec}(\mathcal{A}_1) = \text{Spec}(\mathcal{A}_2)$ .  $\square$

Notice that in the discussion above, the operator  $\mathcal{A}_2$  may be self-adjoint on  $\mathcal{H}_2$ , even though the operator  $\mathcal{A}_1$  need not be self-adjoint on  $\mathcal{H}_1$ . It is therefore natural to ask whether a  $\mathcal{PT}$ -symmetric operator with real spectrum is always similar to a self-adjoint one. This problem has attracted considerable attention, notably by [22], [23], [24], in an abstract framework. Here instead, we shall consider the class of  $\mathcal{PT}$ -symmetric elliptic quadratic differential operators acting on  $L^2(\mathbf{R}^n)$ , with real spectrum, and establish necessary and sufficient conditions for their similarity to a self-adjoint operator, as above. A noticeable peculiarity of these conditions is their classical nature, i.e. their dependence only on the classical flow generated by the classical quadratic hamiltonian, whose quantization yields the given  $\mathcal{PT}$ -symmetric quadratic differential operator. Let us also remark here that in general, the operator  $S$  in (1.1) realizing the similarity between the non-self-adjoint operator  $\mathcal{A}_1$  and the self-adjoint operator  $\mathcal{A}_2$ , cannot be expected to be bounded with a bounded inverse, due to the issue of pseudospectra for non-self-adjoint operators, see [10], [9], [2].

Let  $q(x, \xi)$  be a complex-valued quadratic form on  $\mathbf{R}^{2n} = \mathbf{R}_x^n \times \mathbf{R}_\xi^n$ . Here  $(x, \xi) \in \mathbf{R}^{2n}$  are the canonical coordinates, so that  $\{\xi_i, x_j\} = \delta_{ij}$ ,  $1 \leq i, j \leq n$ . Throughout this work, we shall assume that the quadratic form  $q$  is elliptic on  $\mathbf{R}^{2n}$ , in the sense that

$q(X) = 0$ ,  $X \in \mathbf{R}^{2n}$ , if and only if  $X = 0$ . An application of Lemma 3.1 of [25] then shows, if  $n > 1$ , that there exists  $z \in \mathbf{C} \setminus \{0\}$  such that  $\operatorname{Re}(zq)$  is positive definite. In the case when  $n = 1$ , the same conclusion holds, provided that the range of  $q$  on  $\mathbf{R}^2$  is not all of  $\mathbf{C}$ , which will be assumed in the sequel. After a multiplication by  $z$ , we shall assume in what follows, as we may, that  $z = 1$ , so that  $\operatorname{Re} q$  is positive definite. It follows that the range  $\Sigma(q) := q(\mathbf{R}^{2n})$  of  $q$  on  $\mathbf{R}^{2n}$  is a closed angular sector with a vertex at zero, contained in the union of  $\{0\}$  and the open right half-plane.

We shall now introduce the assumption of the  $\mathcal{PT}$ -symmetry of the quadratic symbol  $q$ . To that end, let  $\kappa : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be linear and such that

$$\kappa^2 = 1. \quad (1.3)$$

Associated to the involution  $\kappa$ , we have the parity operator  $\mathcal{P}$ , given by

$$(\mathcal{P}u)(x) = u(\kappa(x)), \quad (1.4)$$

and the lift of  $\kappa$  to the cotangent space, given by the linear involution,

$$\mathcal{K} : \mathbf{R}^{2n} \rightarrow \mathbf{R}^{2n}, \quad \mathcal{K}(x, \xi) = (\kappa(x), -\kappa^t(\xi)). \quad (1.5)$$

The  $\mathcal{PT}$ -symmetry assumption on the quadratic symbol  $q$  is of the following form,

$$\overline{q \circ \mathcal{K}} = q \iff \overline{q(\kappa(x), -\kappa^t(\xi))} = q(x, \xi), \quad (x, \xi) \in \mathbf{R}^{2n}. \quad (1.6)$$

It follows, in particular, that the sector  $\Sigma(q)$  is symmetric with respect to reflection in the real axis.

Associated to the symbol  $q$  is the corresponding operator  $\operatorname{Op}^w(q)$  acting on  $L^2(\mathbf{R}^n)$ , formally defined as the Weyl quantization of  $q(x, \xi)$ ,

$$\operatorname{Op}^w(q)u(x) = (2\pi)^{-n} \iint_{\mathbf{R}^n \times \mathbf{R}^n} e^{i\xi \cdot (x-y)} q\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \quad (1.7)$$

Writing

$$q(x, \xi) = \sum_{|\alpha+\beta|=2} q_{\alpha,\beta} x^\alpha \xi^\beta, \quad (1.8)$$

we see that the operator  $\operatorname{Op}^w(q)$  is a differential operator of the form

$$\operatorname{Op}^w(q) = \sum_{|\alpha+\beta|=2} q_{\alpha,\beta} \frac{x^\alpha D_x^\beta + D_x^\beta x^\alpha}{2}.$$

Let us recall from [19] that the maximal closed realization of the operator  $\text{Op}^w(q)$ , i.e. the operator on  $L^2(\mathbf{R}^n)$  equipped with the domain

$$\{u \in L^2(\mathbf{R}^n); \text{Op}^w(q)u \in L^2(\mathbf{R}^n)\} = \{u \in L^2(\mathbf{R}^n); x^\alpha D_x^\beta u \in L^2(\mathbf{R}^n), |\alpha + \beta| \leq 2\},$$

agrees with the graph closure of its restriction to  $\mathcal{S}(\mathbf{R}^n)$ ,

$$\text{Op}^w(q) : \mathcal{S}(\mathbf{R}^n) \rightarrow \mathcal{S}(\mathbf{R}^n).$$

Furthermore, the spectrum of  $\text{Op}^w(q)$  is discrete, and its precise description will be recalled below.

The assumption (1.6) implies the  $\mathcal{PT}$ -symmetry of the operator  $\text{Op}^w(q)$ , i.e. the commutation property  $[\text{Op}^w(q), \mathcal{PT}] = 0$ . Here the parity operator  $\mathcal{P}$  has been introduced in (1.4). Indeed, we have

$$\begin{aligned} \mathcal{P} \circ \text{Op}^w(q) \circ \mathcal{P}^{-1} &= \text{Op}^w(q(\kappa(x), \kappa^t(\xi))); \quad \mathcal{T} \circ \text{Op}^w(q) \circ \mathcal{T}^{-1} = \text{Op}^w(\bar{q}(x, -\xi)), \\ \mathcal{PT} \circ \text{Op}^w(q) \circ (\mathcal{PT})^{-1} &= \text{Op}^w(\bar{q}(\kappa(x), -\kappa^t(\xi))). \end{aligned}$$

It follows that the spectrum of  $\text{Op}^w(q)$  is symmetric with respect to the real axis.

The essential role in what follows will be played by the fundamental matrix  $F$  of the quadratic form  $q$ . When recalling the definition of  $F$  following Section 21.5 of [18], we let

$$\sigma((x, \xi), (y, \eta)) = \xi \cdot y - \eta \cdot x, \quad (x, \xi) \in \mathbf{R}^{2n}, \quad (y, \eta) \in \mathbf{R}^{2n},$$

be the canonical symplectic form on  $\mathbf{R}^{2n}$ , which extends to the complex symplectic form on  $\mathbf{C}^{2n}$ . Letting also  $q(X, Y)$  stand for the polarization of  $q$ , viewed as a symmetric bilinear form on  $\mathbf{C}^{2n}$ , we define the  $2n \times 2n$  fundamental matrix  $F$  by the identity

$$q(X, Y) = \sigma(X, FY), \quad X, Y \in \mathbf{C}^{2n}. \quad (1.9)$$

We notice that the fundamental matrix  $F$  is skew-symmetric with respect to  $\sigma$ , and furthermore, following [18], we see that in the canonical coordinates  $(x, \xi)$ , it is given by

$$F = \frac{1}{2} \begin{pmatrix} q''_{\xi, x} & q''_{\xi, \xi} \\ -q''_{x, x} & -q''_{x, \xi} \end{pmatrix}. \quad (1.10)$$

We can now state the main result of this work.

**Theorem 1.2** *Let  $q : \mathbf{R}_x^n \times \mathbf{R}_\xi^n \rightarrow \mathbf{C}$  be a quadratic form, such that  $\operatorname{Re} q$  is positive definite. Let  $\kappa : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be linear and such that  $\kappa^2 = 1$ . Assume the property of the  $\mathcal{PT}$ -symmetry,*

$$\overline{q(\kappa(x), -\kappa^t(\xi))} = q(x, \xi), \quad (x, \xi) \in \mathbf{R}^{2n},$$

*and that the spectrum of the quadratic operator  $\operatorname{Op}^w(q)$  is purely real. Then the operator  $\operatorname{Op}^w(q)$  is similar to a self-adjoint operator, in the sense of the discussion in the beginning of this section, precisely when the fundamental matrix  $F$  of  $q$  has no Jordan blocks.*

**Remarks.**

1. Let us consider the Hamilton vector field of  $q$ ,

$$H_q = q'_\xi \cdot \partial_x - q'_x \cdot \partial_\xi. \quad (1.11)$$

According to (1.10), we have

$$FY = \frac{1}{2}H_q(Y).$$

Therefore the possibility of establishing a similarity between a  $\mathcal{PT}$ -symmetric elliptic quadratic operator with purely real spectrum and a self-adjoint operator involves only the underlying classical flow, i.e. is determined by a purely classical condition.

2. Examples of  $\mathcal{PT}$ -symmetric non-diagonalizable hamiltonians with real spectrum can be found in [7], as well as in [1], [30], [8]. See also the discussion in Section 4.
3. In Theorem 1.2 we have only considered the case when the quadratic form  $q$  is elliptic on  $\mathbf{R}^{2n}$ . It seems quite likely, however, that the result of Theorem 1.2 can be extended to a suitable class of partially elliptic quadratic operators, namely the one studied in [14], [31]. We leave this extension open until needed.

The plan of this note is as follows. In Section 2, we establish a symmetry property of the fundamental matrix  $F$ , as a consequence of the assumption of the  $\mathcal{PT}$ -symmetry of  $q$ . The proof of Theorem 1.2 is then carried out in Section 3, using the techniques of FBI–Bargmann transformations with quadratic phases [26], which, in the quadratic case, allow us to construct the similarity transformation  $S$

explicitly. Section 4 is devoted to the discussion of an example, due to [8], of an elliptic quadratic  $\mathcal{PT}$ -symmetric operator with real spectrum, for which the fundamental matrix possesses Jordan blocks. The appendix A gives a concise exposition of aspects of the theory of positive Lagrangian planes, required in the proof of the main result, while in the appendix B, the general theory of quadratic Fourier integral operators in the complex domain, arising when quantizing complex linear canonical transformations, is developed.

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## 2 $\mathcal{PT}$ -invariance condition for the fundamental matrix

In this section, we let  $q : \mathbf{R}^{2n} \rightarrow \mathbf{C}$  be a quadratic form such that  $\operatorname{Re} q$  is positive definite. It follows therefore from (1.9) that the eigenvalues of the corresponding fundamental matrix  $F$  avoid the real axis, and in general we know from Section 21.5 of [18] that if  $\lambda$  is an eigenvalue of  $F$ , then so is  $-\lambda$ , and the algebraic multiplicities agree. Let us also recall from [18] that the eigenvalues of  $F$  belong to the set  $i\Sigma(q) \cup -i\Sigma(q)$ , and let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues of  $F$ , counted according to their multiplicity, such that  $\lambda_j/i \in \Sigma(q)$ ,  $j = 1, \dots, n$ . From [6], [25], we know that the spectrum of the operator  $\operatorname{Op}^w(q)$  is given by the eigenvalues of the form

$$\sum_{j=1}^n \frac{\lambda_j}{i} (2\nu_{j,\ell} + 1), \quad \nu_{j,\ell} \in \mathbf{N} \cup \{0\}. \quad (2.1)$$

Notice that  $\operatorname{Spec}(\operatorname{Op}^w(q)) \subset \Sigma(q)$ .

Let us recall the real linear involution  $\mathcal{K}$ , introduced in (1.5). Writing  $\mathcal{K} = J \circ \mathcal{K}_1$ , where  $J(x, \xi) = (x, -\xi)$  and  $\mathcal{K}_1(y, \eta) = (\kappa(y), (\kappa^t)^{-1}(\eta))$  is symplectic, we see, using also that the involution  $J$  is skew-symmetric with respect to the symplectic form  $\sigma$ , that the map  $\mathcal{K}$  is antisymplectic in the sense that

$$\sigma(\mathcal{K}X, \mathcal{K}Y) = -\sigma(X, Y), \quad X, Y \in \mathbf{R}^{2n}.$$

In particular,

$$\sigma(\mathcal{K}X, Y) = -\sigma(\mathcal{K}^2 X, \mathcal{K}Y) = -\sigma(X, \mathcal{K}Y).$$

We shall also let  $\mathcal{C} : \mathbf{C}^{2n} \rightarrow \mathbf{C}^{2n}$  stand for the involution defined by the operation of complex conjugation. We have the following result.

**Proposition 2.1** *Assume that the quadratic form  $q$  with  $\operatorname{Re} q > 0$  is such that  $\overline{q(\mathcal{K}(x, \xi))} = q(x, \xi)$ ,  $(x, \xi) \in \mathbf{R}^{2n}$ . Then the fundamental matrix  $F$  of  $q$  satisfies*

$$F = -\mathcal{K}\mathcal{C} \circ F \circ \mathcal{K}\mathcal{C}. \quad (2.2)$$

*Furthermore, the eigenvalues of  $F$  are symmetric with respect to the imaginary axis.*

*Proof:* Let us write,

$$q(x, \xi) = Ax \cdot x + 2Bx \cdot \xi + C\xi \cdot \xi.$$

Here  $A$ ,  $B$ , and  $C$  are complex  $n \times n$  matrices, with  $A$  and  $C$  symmetric. The condition of the  $\mathcal{PT}$ -symmetry,

$$\overline{q(\kappa(x), -\kappa^t(\xi))} = q(x, \xi), \quad (x, \xi) \in \mathbf{R}^{2n},$$

implies then that the matrices  $A$ ,  $B$ , and  $C$  satisfy

$$\overline{A} = \kappa^t A \kappa, \quad \overline{B} = -\kappa B \kappa, \quad \overline{C} = \kappa C \kappa^t. \quad (2.3)$$

When establishing the property (2.2), we introduce the symmetric bilinear form associated to  $q$ , given by

$$q((x, \xi), (y, \eta)) = Ax \cdot y + Bx \cdot \eta + By \cdot \xi + C\xi \cdot \eta, \quad (x, \xi) \in \mathbf{C}^{2n}, \quad (y, \eta) \in \mathbf{C}^{2n}.$$

Using (2.3), we see that

$$\overline{q(\mathcal{K}(x, \xi), \mathcal{K}(y, \eta))} = q(\mathcal{C}(x, \xi), \mathcal{C}(y, \eta)),$$

and therefore, from (1.9) we get

$$\sigma(\mathcal{C}X, F\mathcal{C}Y) = \overline{\sigma(\mathcal{K}X, F\mathcal{K}Y)} = -\overline{\sigma(X, \mathcal{K}F\mathcal{K}Y)} = -\sigma(\mathcal{C}X, \mathcal{C}\mathcal{K}F\mathcal{K}Y).$$

Here we have also used that the map  $\mathcal{K}$  is skew-symmetric with respect to  $\sigma$ . The identity (2.2) follows, and it only remains to check the symmetry of the eigenvalues of  $F$  with respect to the imaginary axis. To that end, assume that  $\lambda \in \mathbf{C}$  is such that  $FX = \lambda X$ ,  $X \in \mathbf{C}^{2n}$ . If  $Y = \mathcal{K}\mathcal{C}X$ , then  $(F \circ \mathcal{K}\mathcal{C})Y = \lambda\mathcal{K}\mathcal{C}Y$ . Applying the antilinear map  $\mathcal{K}\mathcal{C}$  and using (2.2), we conclude that  $FY = -\overline{\lambda}Y$ . The proof is complete.  $\square$

**Remark 2.2.** Combining Proposition 2.1 with the explicit description of the spectrum of the  $\mathcal{PT}$ -symmetric operator  $\operatorname{Op}^w(q)$ , given by (2.1), we conclude that if the spectrum is purely real, then the eigenvalues  $\lambda_j$ ,  $1 \leq j \leq n$ , are purely imaginary.



### 3 Similarity transformation and proof of Theorem 1.2

Assume that  $q : \mathbf{R}^{2n} \rightarrow \mathbf{C}$  is a quadratic form such that  $\operatorname{Re} q > 0$ , and let  $F$  be the fundamental matrix of  $q$ , introduced in (1.9). When  $\lambda \in \operatorname{Spec}(F)$ , we let

$$E_\lambda = \operatorname{Ker}((F - \lambda)^{2n}) \subset \mathbf{C}^{2n} \quad (3.1)$$

be the spectral subspace corresponding to the eigenvalue  $\lambda$ . According to Lemma 21.5.2 of [18], the complex symplectic form  $\sigma$  is nondegenerate viewed as a bilinear form on  $E_\lambda \times E_{-\lambda}$ .

Let us introduce the unstable linear manifold for the Hamilton flow of the quadratic form  $i^{-1}q$ , given by

$$\Lambda^+ := \bigoplus_{\operatorname{Im} \lambda > 0} E_\lambda \subset \mathbf{C}^{2n}. \quad (3.2)$$

According to Proposition 3.3 of [25], the complex Lagrangian plane  $\Lambda^+$  is strictly positive in the sense that

$$\frac{1}{i} \sigma(X, \overline{X}) > 0, \quad 0 \neq X \in \Lambda^+. \quad (3.3)$$

We refer to Appendix A for a discussion of positivity conditions in  $\mathbf{C}^{2n}$ . Defining also

$$\Lambda^- = \bigoplus_{\operatorname{Im} \lambda < 0} E_\lambda \subset \mathbf{C}^{2n}, \quad (3.4)$$

we know from the arguments of [25] that the complex Lagrangian plane  $\Lambda^-$  is strictly negative in the sense that

$$\frac{1}{i} \sigma(X, \overline{X}) < 0, \quad 0 \neq X \in \Lambda^-. \quad (3.5)$$

It also follows from (1.9) that the quadratic form  $q$  vanishes when restricted to  $\Lambda^\pm$ .

Our proof of Theorem 1.2 will proceed by exhibiting a complex linear canonical transformation which will reduce  $\Lambda^+$  to  $\{(x, \xi) \in \mathbf{C}^{2n}; \xi = 0\}$  and  $\Lambda^-$  to  $\{(x, \xi) \in \mathbf{C}^{2n}; x = 0\}$ . We shall then be able to quantize the canonical transformation by means of an FBI–Bargmann transform [26], which will essentially provide us with the sought similarity operator. Let us notice here that the use of such canonical transformations has a long and rich tradition, see for instance [21], [12], [28]. The discussion below will follow Section 2 of [15] closely.

Although not necessary, it will be convenient to simplify  $q$  first by means of a suitable real linear canonical transformation. When doing so, we observe that an application of Example A.6 in Appendix A shows that the negative Lagrangian  $\Lambda^-$  is of the form

$$\eta = A_- y, \quad y \in \mathbf{C}^n,$$

where the complex symmetric  $n \times n$  matrix  $A_-$  is such that  $\operatorname{Im} A_- < 0$ . Here  $(y, \eta)$  are the standard canonical coordinates on  $T^*\mathbf{R}_y^n = \mathbf{R}_y^n \times \mathbf{R}_\eta^n$ , extended to the complexification  $T^*\mathbf{C}_y^n$ . Using the real linear canonical transformation  $(y, \eta) \mapsto (y, \eta - (\operatorname{Re} A_-)y)$ , we reduce  $\Lambda^-$  to the form  $\eta = i\operatorname{Im} A_- y$ , and by a diagonalization of  $\operatorname{Im} A_-$ , we obtain the standard form  $\eta = -iy$ . After this real linear canonical transformation and the conjugation of the quadratic operator  $\operatorname{Op}^w(q)$  by means of the corresponding metaplectic operator, unitary on  $L^2(\mathbf{R}^n)$ , we may assume that  $\Lambda^-$  is of the form

$$\eta = -iy, \quad y \in \mathbf{C}^n. \quad (3.6)$$

We refer to the Appendix to Chapter 7 of [11] for the introduction to and basic properties of metaplectic operators. Notice also that according to Example A.6, in the new real symplectic coordinates, extended to the complexification, the positive complex Lagrangian  $\Lambda^+$  is of the form

$$\eta = A_+ y, \quad \operatorname{Im} A_+ > 0. \quad (3.7)$$

Let

$$B = B_+ = (1 - iA_+)^{-1}A_+, \quad (3.8)$$

and notice that the matrix  $B$  is symmetric. The holomorphic quadratic form

$$\varphi(x, y) = \frac{i}{2}(x - y)^2 - \frac{1}{2}Bx \cdot x, \quad (x, y) \in \mathbf{C}^{2n}, \quad (3.9)$$

satisfies  $\operatorname{Im} \varphi''_{y,y} > 0$  and  $\det \varphi''_{x,y} \neq 0$ . It gives rise therefore to the FBI–Bargmann transformation,

$$Tu(x) = C \int_{\mathbf{R}^n} e^{i\varphi(x,y)} u(y) dy, \quad x \in \mathbf{C}^n, \quad C > 0, \quad (3.10)$$

and following [26], [29], the operator  $T$  is to be viewed as a Fourier integral operator, with the associated complex linear canonical transformation of the form

$$\kappa_T : \mathbf{C}^{2n} \ni (y, -\varphi'_y(x, y)) \mapsto (x, \varphi'_x(x, y)) \in \mathbf{C}^{2n}. \quad (3.11)$$

Using (3.8), (3.9), we see that  $\kappa_T$  is given by

$$\kappa_T : (y, \eta) \mapsto (x, \xi) = (y - i\eta, \eta + iB\eta - By), \quad (3.12)$$

and therefore, we have  $\kappa_T(\Lambda_-) = \{(x, \xi) \in \mathbf{C}^{2n}; x = 0\}$ , while  $\kappa_T(\Lambda^+)$  is given by the equation  $\{(x, \xi) \in \mathbf{C}^{2n}; \xi = 0\}$ .

We know from [29] that for a suitable choice of  $C > 0$  in (3.10), the map  $T$  is unitary,

$$T : L^2(\mathbf{R}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n), \quad (3.13)$$

where

$$H_{\Phi_0}(\mathbf{C}^n) = \text{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n; e^{-2\Phi_0} L(dx)),$$

and  $\Phi_0$  is a strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$ , given by

$$\Phi_0(x) = \sup_{y \in \mathbf{R}^n} (-\text{Im } \varphi(x, y)) = \frac{1}{2} ((\text{Im } x)^2 + \text{Im } (Bx \cdot x)). \quad (3.14)$$

From [29], we recall also that the canonical transformation  $\kappa_T$  in (3.11) maps  $\mathbf{R}^{2n}$  bijectively onto

$$\Lambda_{\Phi_0} = \left\{ \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right); x \in \mathbf{C}^n \right\}. \quad (3.15)$$

The positivity of  $\{(x, \xi) \in \mathbf{C}^{2n}; \xi = 0\}$  with respect to  $\Lambda_{\Phi_0}$  implies, in view of Proposition A.9 in Appendix A, that the quadratic form  $\Phi_0$  is strictly convex, so that

$$\Phi_0(x) \sim |x|^2, \quad x \in \mathbf{C}^n. \quad (3.16)$$

**Example: The standard Bargmann transformation.** Following [3], let us consider a complex integral transform of the form (3.10), where the phase function  $\varphi(x, y)$  is given by

$$\varphi(x, y) = i \left( \frac{x^2}{2} + \sqrt{2}xy + \frac{y^2}{2} \right).$$

The corresponding canonical transformation, defined as in (3.11), is then of the form

$$(y, \eta) \mapsto \frac{1}{\sqrt{2}}(i\eta - y, -\eta + iy),$$

while the associated quadratic weight function is given by  $\Phi(x) = (1/2) |x|^2$ .

Returning to the FBI–Bargmann transformation given by (3.9), (3.10), let us next recall the exact Egorov theorem, [29],

$$T\text{Op}^w(q)u = \text{Op}^w(\tilde{q})Tu, \quad u \in \mathcal{S}(\mathbf{R}^n), \quad (3.17)$$

where  $\tilde{q}$  is a quadratic form on  $\mathbf{C}^{2n}$  given by  $\tilde{q} = q \circ \kappa_T^{-1}$ . It follows therefore that

$$\tilde{q}(x, \xi) = Mx \cdot \xi, \quad (3.18)$$

where  $M$  is a complex  $n \times n$  matrix. We have

$$H_{\tilde{q}} = Mx \cdot \partial_x - M^t \xi \cdot \partial_\xi,$$

and using (1.10), we see that the corresponding Hamilton map

$$\tilde{F} = \frac{1}{2} \begin{pmatrix} M & 0 \\ 0 & -M^t \end{pmatrix}$$

maps  $(x, 0) \in \kappa_T(\Lambda^+)$  to  $(1/2)(Mx, 0)$ . Now the maps  $F$  and  $\tilde{F}$  are isospectral, and we conclude that, with the agreement of algebraic multiplicities, the following holds,

$$\text{Spec}(M) = \text{Spec}(2F) \cap \{\text{Im } \lambda > 0\}. \quad (3.19)$$

The final step in the normal form construction is the reduction of the matrix  $M$  in (3.18) to its Jordan normal form. Such a reduction is implemented by considering a complex linear canonical transformation of the form

$$\kappa_C : \mathbf{C}^{2n} \ni (x, \xi) \mapsto (C^{-1}x, C^t \xi) \in \mathbf{C}^{2n}, \quad (3.20)$$

where  $C$  is a suitable invertible complex  $n \times n$  matrix. On the operator level, associated to the transformation in (3.20), we have the operator

$$U_C : u(x) \mapsto |\det C| u(Cx), \quad (3.21)$$

which maps the space  $H_{\Phi_0}(\mathbf{C}^n)$  unitarily onto the space  $H_{\Phi_1}(\mathbf{C}^n)$ , where  $\Phi_1(x) = \Phi_0(Cx)$  is a strictly plurisubharmonic quadratic form such that  $\kappa_C(\Lambda_{\Phi_0}) = \Lambda_{\Phi_1}$ . We notice that the strict convexity property

$$\Phi_1(x) \sim |x|^2, \quad x \in \mathbf{C}^n, \quad (3.22)$$

remains valid.

We obtain therefore the following result, which only is a slight reformulation of Proposition 2.1 of [15] and is closely related to the discussion in Section 3 of [25].

**Proposition 3.1** *Let  $q : \mathbf{R}^{2n} \rightarrow \mathbf{C}$  be a quadratic form, such that  $\operatorname{Re} q > 0$ . The quadratic differential operator*

$$\operatorname{Op}^w(q) : L^2(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n),$$

*equipped with the domain*

$$\mathcal{D}(\operatorname{Op}^w(q)) = \{u \in L^2(\mathbf{R}^n); (x^2 + (D_x)^2) u \in L^2(\mathbf{R}^n)\},$$

*is unitarily equivalent to the quadratic operator,*

$$\operatorname{Op}^w(\tilde{q}) : H_{\Phi_1}(\mathbf{C}^n) \rightarrow H_{\Phi_1}(\mathbf{C}^n),$$

*with the domain*

$$\mathcal{D}(\operatorname{Op}^w(\tilde{q})) = \{u \in H_{\Phi_1}(\mathbf{C}^n); (1 + |x|^2)u \in L^2_{\Phi_1}(\mathbf{C}^n)\}.$$

*Here*

$$\tilde{q}(x, \xi) = Mx \cdot \xi,$$

*where  $M$  is a complex  $n \times n$  block-diagonal matrix, each block being a Jordan one. Furthermore, the eigenvalues of  $M$  are precisely those of  $2F$  in the upper half-plane, and the quadratic weight function  $\Phi_1(x)$  satisfies,*

$$\Phi_1(x) \sim |x|^2, \quad x \in \mathbf{C}^n.$$

*The unitary equivalence between the operators  $\operatorname{Op}^w(q)$  and  $\operatorname{Op}^w(\tilde{q})$  is realized by an operator of the form  $U_1 \circ T \circ U_2 : L^2(\mathbf{R}^n) \rightarrow H_{\Phi_1}(\mathbf{C}^n)$ , where  $U_2$  is a unitary metaplectic operator on  $L^2(\mathbf{R}^n)$ , while  $T$  is an FBI–Bargmann transform of the form (3.9), (3.10), and  $U_1$  is an operator of the form (3.21).*

Let us recall now that  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of the fundamental matrix  $F$  of  $q$  in the upper half-plane, repeated according to their multiplicity. It follows from Proposition 3.1 that the action of the operator  $\operatorname{Op}^w(\tilde{q})$  on  $H_{\Phi_1}(\mathbf{C}^n)$  is given by

$$\operatorname{Op}^w(\tilde{q}) = \sum_{j=1}^n 2\lambda_j x_j D_{x_j} + \frac{1}{i} \sum_{j=1}^n \lambda_j + \sum_{j=1}^{n-1} \gamma_j x_{j+1} D_{x_j}, \quad \gamma_j \in \{0, 1\}. \quad (3.23)$$

Let

$$\Phi(x) = (1/2) |x|^2, \quad x \in \mathbf{C}^n,$$

be the standard radial weight function. We observe that the weighted spaces  $H_{\Phi_1}(\mathbf{C}^n)$  and  $H_{\Phi}(\mathbf{C}^n)$  contain a common dense subset, namely the space of holomorphic polynomials on  $\mathbf{C}^n$ . Indeed, it is well known that the normalized monomials form an orthonormal basis in  $H_{\Phi}(\mathbf{C}^n)$ , while the density of holomorphic polynomials in  $H_{\Phi_1}(\mathbf{C}^n)$  has been explained in [15], and follows from the fact that the finite linear combinations of the generalized eigenfunctions of  $\text{Op}^w(q)$  are dense in  $L^2(\mathbf{R}^n)$ . When restricted to the space of holomorphic polynomials, the action of the operator  $\text{Op}^w(\tilde{q})$  on  $H_{\Phi_1}(\mathbf{C}^n)$  is clearly similar to the action of the closed densely defined operator  $\text{Op}^w(\tilde{q})$  on the space  $H_{\Phi}(\mathbf{C}^n)$ , the corresponding unbounded densely defined similarity transformation  $S : H_{\Phi_1}(\mathbf{C}^n) \rightarrow H_{\Phi}(\mathbf{C}^n)$  being the identity map, with  $\mathcal{D}(S)$  being the space of holomorphic polynomials on  $\mathbf{C}^n$ .

Assume now that the  $\mathcal{PT}$ -symmetry condition,  $\overline{q(\kappa(x), -\kappa^t(\xi))} = q(x, \xi)$ ,  $(x, \xi) \in \mathbf{R}^{2n}$ , is maintained, and that the spectrum of the quadratic operator  $\text{Op}^w(q)$  on  $L^2(\mathbf{R}^n)$  is purely real. According to Remark 2.2, we then know that the eigenvalues  $\lambda_j$ ,  $1 \leq j \leq n$ , are purely imaginary. If furthermore the nilpotent part in the Jordan decomposition of  $F$  vanishes, then (3.23) shows that

$$\text{Op}^w(\tilde{q}) = \sum_{j=1}^n 2\lambda_j x_j D_{x_j} + \frac{1}{i} \sum_{j=1}^n \lambda_j, \quad (3.24)$$

which, when equipped with the domain,

$$\mathcal{D}(\text{Op}^w(\tilde{q})) = \{u \in H_{\Phi}(\mathbf{C}^n); (1 + |x|^2)u \in L^2_{\Phi}(\mathbf{C}^n)\},$$

is seen to be self-adjoint in the Bargmann space  $H_{\Phi}(\mathbf{C}^n)$ , by means of a direct computation. This establishes the sufficiency part in Theorem 1.2.

It only remains to verify the necessity in Theorem 1.2. To this end, assume that the  $\mathcal{PT}$ -symmetric elliptic quadratic operator  $\text{Op}^w(q)$ ,  $\text{Re } q > 0$ , is similar to a self-adjoint operator, in the sense of the discussion in Section 1. It follows that for each  $\lambda \in \text{Spec}(\text{Op}^w(q))$ , the corresponding spectral subspace  $E_{\lambda} \subset \mathcal{S}(\mathbf{R}^n)$  consists entirely of eigenvectors. We conclude then from Proposition 3.1 that the nilpotent part in the Jordan decomposition of  $F$  vanishes. The proof of Theorem 1.2 is complete.

**Remark 3.3.** If we drop the  $\mathcal{PT}$ -symmetry assumption and merely assume that there are no Jordan blocks in the Jordan normal form of the fundamental matrix  $F$ , then it follows from (3.24) that the operator  $\text{Op}^w(\tilde{q})$ , acting on  $H_{\Phi}(\mathbf{C}^n)$ , is normal, and therefore, the original quadratic operator  $\text{Op}^w(q)$  acting on  $L^2(\mathbf{R}^n)$  is similar to a normal operator.

## 4 Example: quantum harmonic oscillator with quadratic complex interaction

The purpose of this section is to illustrate Theorem 1.2 by applying it to the following two-dimensional quadratic Schrödinger operator,

$$\text{Op}^w(q) = D_{x_1}^2 + D_{x_2}^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + 2igx_1x_2. \quad (4.1)$$

Here  $\omega_j > 0$ ,  $j = 1, 2$ ,  $\omega_1 \neq \omega_2$ , and  $g \in \mathbf{R}$ . The operator  $\text{Op}^w(q)$  has been considered in the work [8] as a quantum model of a non-isotropic two-dimensional harmonic oscillator, perturbed by an additional quadratic interaction, with a purely imaginary coupling constant.

The quadratic operator  $\text{Op}^w(q)$  is globally elliptic and  $\mathcal{PT}$ -symmetric, with the corresponding involution given by  $\kappa(x_1, x_2) = (-x_1, x_2)$ . In [8], the authors show that a certain method of separation of variables fails to work for  $\text{Op}^w(q)$  precisely when

$$2g = \pm(\omega_1^2 - \omega_2^2). \quad (4.2)$$

Furthermore, in the case when (4.2) holds, it is shown in [8] that the eigenfunctions of  $\text{Op}^w(q)$  do not form a complete set in  $L^2(\mathbf{R}^2)$ .

By applying Theorem 1.2, here we shall show the following result.

**Proposition 4.1** *The spectrum of  $\text{Op}^w(q)$  is real precisely when*

$$-|\omega_1^2 - \omega_2^2| \leq 2g \leq |\omega_1^2 - \omega_2^2|, \quad (4.3)$$

*and  $\text{Op}^w(q)$  is similar to a self-adjoint operator if and only if*

$$-|\omega_1^2 - \omega_2^2| < 2g < |\omega_1^2 - \omega_2^2|. \quad (4.4)$$

*Proof:* The Weyl symbol  $q$  of the operator  $\text{Op}^w(q)$  is

$$q(x, \xi) = \sum_{j=1}^2 (\xi_j^2 + \omega_j^2 x_j^2) + 2igx_1x_2, \quad (4.5)$$

and the corresponding fundamental matrix  $F$  is given by

$$F = \frac{1}{2} \begin{pmatrix} q''_{\xi,x} & q''_{\xi,\xi} \\ -q''_{x,x} & -q''_{x,\xi} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & -ig & 0 & 0 \\ -ig & -\omega_2^2 & 0 & 0 \end{pmatrix}. \quad (4.6)$$

A straightforward computation then shows that

$$\det(F - \lambda I) = \lambda^4 + (\omega_1^2 + \omega_2^2)\lambda^2 + \omega_1^2\omega_2^2 + g^2, \quad (4.7)$$

with the four eigenvalues  $\lambda$  being given by

$$\lambda^2 = -\frac{\omega_1^2 + \omega_2^2}{2} \pm \sqrt{\left(\frac{\omega_1^2 - \omega_2^2}{2}\right)^2 - g^2}. \quad (4.8)$$

The expression under the square root sign is strictly less than  $((\omega_1^2 + \omega_2^2)/2)^2$ , so that the eigenvalues are non-vanishing, as we already know since  $q$  is elliptic. It vanishes precisely when (4.2) holds, and from (4.8) we conclude that the eigenvalues are simple precisely when (4.2) does not hold. When (4.2) holds, then the spectrum of  $F$  consists of two double eigenvalues,

$$\lambda_1 = \lambda_2 = i \left( \frac{\omega_1^2 + \omega_2^2}{2} \right)^{1/2}, \quad \lambda_3 = \lambda_4 = -i \left( \frac{\omega_1^2 + \omega_2^2}{2} \right)^{1/2}. \quad (4.9)$$

When (4.3) holds, we see that the eigenvalues  $\lambda$  of  $F$  are on the imaginary axis, so thanks to (2.1), we know that the spectrum of  $\text{Op}^w(q)$  is real. When (4.3) does not hold, the square root in (4.8) is non-real, so the eigenvalues  $\lambda$  are off the imaginary axis, and  $\text{Op}^w(q)$  has spectrum away from  $\mathbf{R}$ , as was observed in Remark 2.2.

From the discussion so far, we get the first statement in the proposition concerning the reality of the spectrum of  $\text{Op}^w(q)$ , and also the fact that (4.4) is a sufficient condition for  $\text{Op}^w(q)$  to be similar to a self-adjoint operator. It only remains therefore to show that  $\text{Op}^w(q)$  is not similar to a self-adjoint operator when  $2g = \pm(\omega_1^2 - \omega_2^2)$ , and in view of Theorem 1.2, it suffices to show that Jordan blocks do occur in  $F$  then. In this case, we have two eigenvalues,

$$\lambda_{\pm} = \pm i \left( \frac{\omega_1^2 + \omega_2^2}{2} \right)^{1/2},$$

each of algebraic multiplicity two. If no Jordan block is present in the spectral subspace of  $\lambda_+$ , say, then the rank of  $F - \lambda_+ I$  is equal to two. On the other hand, we have, using (4.6),

$$F - \lambda_+ I = \begin{pmatrix} -\lambda_+ & 0 & 1 & 0 \\ 0 & -\lambda_+ & 0 & 1 \\ -\omega_1^2 & -ig & -\lambda_+ & 0 \\ -ig & -\omega_2^2 & 0 & -\lambda_+ \end{pmatrix}, \quad (4.10)$$



and the last three columns are linearly independent, since,

$$\det \begin{pmatrix} -\lambda_+ & 0 & 1 \\ -ig & -\lambda_+ & 0 \\ -\omega_2^2 & 0 & -\lambda_+ \end{pmatrix} = -\lambda_+(\lambda_+^2 + \omega_2^2) \neq 0.$$

The proof is complete.  $\square$

## A Positivity

The purpose of this appendix is to provide a brief but complete account of the relevant aspects of the theory of positive complex Lagrangian planes, required in the proof of Theorem 1.2. See also Chapter 11 of [26], where a more general discussion is given.

Let  $\Sigma \subset \mathbf{C}^{2n}$  be a real subspace of dimension  $2n$  which is symplectic in the sense that  $\sigma|_\Sigma$  is a real and nondegenerate 2-form, where  $\sigma = \sum_1^n d\xi_j \wedge dx_j$  is the complex symplectic 2-form on  $\mathbf{C}^{2n} = \mathbf{C}_x^n \times \mathbf{C}_\xi^n$ . Then  $\Sigma$  is maximally totally real in the sense that  $\Sigma \cap i(\Sigma) = 0$  and the real dimension  $\dim_{\mathbf{R}}(\Sigma)$  is maximal ( $2n$ ).

Let  $\iota = \iota_\Sigma : \mathbf{C}^{2n} \rightarrow \mathbf{C}^{2n}$  be the unique anti-linear map which is equal to the identity on  $\Sigma$ . Clearly,

$$\iota^* \sigma = \bar{\sigma}.$$

**Example A.1.**  $\Sigma = \mathbf{R}^{2n}$ ,  $\iota(\rho) = \bar{\rho}$  is the usual complex conjugation.

**Example A.2.** Let  $\Phi = \Phi(x)$  be a real quadratic form on  $\mathbf{C}^n$  such that the Levi matrix  $\partial_{\bar{x}} \partial_x \Phi$  is nondegenerate. Put

$$\Sigma = \Lambda_\Phi = \left\{ \left( x, \frac{2}{i} \frac{\partial \Phi}{\partial x}(x) \right); x \in \mathbf{C}^n \right\}. \quad (\text{A.1})$$

Then  $\Sigma$  is symplectic. In fact, using  $x$  to parametrize  $\Sigma$  we get by a straightforward computation

$$\sigma|_{\Lambda_\Phi} = \sum_{j=1}^n \sum_{k=1}^n \frac{2}{i} \frac{\partial^2 \Phi}{\partial \bar{x}_j \partial x_k} d\bar{x}_j \wedge dx_k.$$

Using only that  $\Phi$  is real, we see that this restriction is real. Hence  $\Sigma$  is an I-Lagrangian manifold, i.e. a Lagrangian manifold for the real symplectic form  $\text{Im } \sigma$ . When the Levi form is nondegenerate, we see from the above expression that  $\sigma|_\Sigma$  is nondegenerate.

Let  $\Psi(x, y)$  be the unique holomorphic quadratic form on  $\mathbf{C}_{x,y}^{2n}$  such that

$$\Psi(x, \bar{x}) = \Phi(x), \quad x \in \mathbf{C}^n.$$

Differentiating this relation, we get

$$\partial_x^2 \Psi = \partial_x^2 \Phi, \quad \partial_x \partial_y \Psi = \partial_x \partial_{\bar{x}} \Phi, \quad \partial_y^2 \Psi = \partial_{\bar{x}}^2 \Phi,$$

keeping in mind that all second derivatives of  $\Psi, \Phi$  are constant.

The involution  $\iota = \iota_{\Lambda_\Phi}$  is given by

$$\iota : \left( y, \frac{2}{i} \frac{\partial \Psi}{\partial y}(x, \bar{y}) \right) \mapsto \left( x, \frac{2}{i} \frac{\partial \Psi}{\partial x}(x, \bar{y}) \right),$$

or more explicitly by

$$\iota : \left( y, \frac{2}{i} (\Phi''_{x,\bar{x}} \bar{x} + \Phi''_{x,x} y) \right) \mapsto \left( x, \frac{2}{i} (\Phi''_{x,x} x + \Phi''_{x,\bar{x}} \bar{y}) \right). \quad (\text{A.2})$$

Let  $\Lambda \subset \mathbf{C}^{2n}$  be a  $\mathbf{C}$ -Lagrangian space, i.e. a complex  $n$ -dimensional subspace such that  $\sigma|_\Lambda = 0$ . If  $\Sigma \subset \mathbf{C}^{2n}$  is a real symplectic subspace of maximal dimension with associated involution  $\iota = \iota_\Lambda$ , we consider the Hermitian bilinear form

$$b(\rho, \mu) = \frac{1}{i} \sigma(\rho, \iota(\mu)) \quad \text{on} \quad \Lambda \times \Lambda. \quad (\text{A.3})$$

The form (A.3) was introduced by L. Hörmander [16], in the case when  $\Sigma = \mathbf{R}^{2n}$ .

**Proposition A.3.** *The form (A.3) is nondegenerate if and only if  $\Lambda \cap \Sigma = \{0\}$ .*

*Proof:* If  $0 \neq \rho \in \Lambda$  belongs to the radical of  $b$ , then  $\iota(\rho)$  is symplectically orthogonal to  $\Lambda$ , and consequently,  $\iota(\rho) \in \Lambda$ . The vectors  $\frac{1}{2}(\rho + \iota(\rho))$  and  $\frac{1}{2i}(\rho - \iota(\rho))$  belong to  $\Sigma \cap \Lambda$  and at least one of them is  $\neq 0$ , so  $\Sigma \cap \Lambda \neq \{0\}$ . Conversely  $\Sigma \cap \Lambda$  is contained in the radical of  $b$ , so we have shown that the radical of  $b$  is non-zero precisely when  $\Sigma \cap \Lambda \neq \{0\}$ .  $\square$

**Example A.4.** If  $\Sigma = \mathbf{R}^{2n}$  and  $\Lambda$  is transversal to the fiber  $F = \{(0, \xi); \xi \in \mathbf{C}^n\}$ , then  $\Lambda = \Lambda_\phi$ :  $\xi = \phi'(x) = \phi''x$ , where  $\phi$  is a complex holomorphic quadratic form on  $\mathbf{C}^n$ . For  $(x, \phi'(x)) = (x, \phi''x) \in \Lambda_\phi$  we get

$$\frac{1}{2i} \sigma(\rho, \bar{\rho}) = (\text{Im } \phi'')x \cdot \bar{x},$$

where

$$\operatorname{Im} \phi'' = \frac{1}{2i}(\phi'' - (\phi'')^*),$$

so the signature of the form (A.3) is equal to that of  $\operatorname{Im} \phi''$ .

**Definition A.5.** Let  $\Lambda \subset \mathbf{C}^{2n}$  be a  $\mathbf{C}$ -Lagrangian space and let  $\Sigma \subset \mathbf{C}^{2n}$  be a maximal real symplectic subspace with the corresponding involution  $\iota$ . We say that  $\Lambda$  is  $\Sigma$ -positive (or  $\Sigma$ -negative), if the Hermitian form (A.3) is positive (respectively negative) definite.

**Example A.6.** If  $\Sigma = \mathbf{R}^{2n}$  then  $\Lambda$  is  $\Sigma$ -positive if and only if  $\Lambda = \Lambda_\phi$ , where  $\operatorname{Im} \phi'' > 0$ .

*Proof:* If  $\Lambda = \Lambda_\phi$  we already know by Example A.4, that  $\Sigma$ -positivity of  $\Lambda$  is equivalent to that of  $\operatorname{Im} \phi''$ . This proves the if-part of the proposition. Conversely, if  $\Lambda$  is  $\Sigma$ -positive, then we see that  $\Lambda$  is transversal to the fiber  $F$ , so  $\Lambda = \Lambda_\phi$  and Example A.4 applies again.  $\square$

The following result is a corollary of Proposition A.3 and Example A.4.

**Corollary A.7** *Let  $\Sigma$  be a fixed maximal real symplectic subspace of  $\mathbf{C}^{2n}$ . Then the set of all  $\Sigma$ -positive  $\mathbf{C}$ -Lagrangian spaces is a connected component in the set of all  $\mathbf{C}$ -Lagrangian spaces that are transversal to  $\Sigma$ .*

In fact, after applying a suitable complex canonical transformation, we may assume that  $\Sigma = \mathbf{R}^{2n}$  and we see from Example A.6 that the set of all  $\Sigma$ -positive  $\mathbf{C}$ -Lagrangian spaces is connected. Proposition A.3 then shows that it is a connected component in the set of all  $\mathbf{C}$ -Lagrangian spaces that are transversal to  $\Sigma$ .

Now, let  $\Sigma = \Lambda_\Phi$  be as in (A.1), where

$$\partial_{\bar{x}} \partial_x \Phi > 0. \tag{A.4}$$

**Proposition A.8.** *The fiber  $F = \{(0, \eta); \eta \in \mathbf{C}^n\}$  is  $\Sigma$ -negative.*

*Proof:* Using (A.2) we see that  $\iota(0, \eta) = (x, \xi)$  is given by

$$\xi = \frac{2}{i} \Phi''_{x,x} x, \quad x = \frac{1}{2i} (\Phi''_{\bar{x},x})^{-1} \bar{\eta}. \tag{A.5}$$

It follows that

$$\frac{1}{i} \sigma(0, \eta; x, \xi) = -\frac{1}{2} \eta \cdot (\Phi''_{\bar{x},x})^{-1} \bar{\eta} \leq -\frac{1}{C} |\eta|^2.$$

$\square$

As we saw in (A.5), the  $\Sigma$ -positive space  $\iota(F)$  is given by

$$\xi = \frac{2}{i}\Phi''_{x,x}x = \frac{\partial}{\partial x} \left( \frac{2}{i}\Phi''_{x,x}x \cdot x \right). \quad (\text{A.6})$$

Here, we notice that  $\Phi(x) = \Phi_{\text{plh}}(x) + \Phi_{\text{herm}}(x)$ , where

$$\Phi_{\text{plh}}(x) = \frac{1}{2}\Phi''_{x,x}x \cdot x + \frac{1}{2}\Phi''_{\bar{x},\bar{x}}\bar{x} \cdot \bar{x} = \frac{1}{2}(\Phi(x) - \Phi(ix))$$

is the pluriharmonic (plh) part and

$$\Phi_{\text{herm}}(x) = \Phi''_{\bar{x},x}x \cdot \bar{x} = \frac{1}{2}(\Phi(x) + \Phi(ix))$$

is the (positive definite) Hermitian part. We conclude that the positive  $\mathbf{C}$ -Lagrangian space  $\iota(F)$  is of the form  $\iota(F) = \Lambda_{\Phi_{\text{plh}}}$ , where  $\Phi(x) - \Phi_{\text{plh}}(x) \asymp |x|^2$ .

**Proposition A.9.** *Let  $\Sigma = \Lambda_{\Phi}$  be as in (A.1), (A.4). A  $\mathbf{C}$ -Lagrangian space is  $\Sigma$ -positive if and only if  $\Lambda = \Lambda_{\tilde{\Phi}}$ , where  $\tilde{\Phi}$  is pluriharmonic and  $\Phi - \tilde{\Phi} \asymp |x|^2$ .*

*Proof:* If we decompose the pluriharmonic form  $\tilde{\Phi}$  as  $\tilde{\Phi} = \Phi_{\text{plh}} + \hat{\Phi}$ , we see that  $\Phi - \tilde{\Phi} > 0$  precisely when  $|\hat{\Phi}(x)| < \Phi_{\text{herm}}(x)$ ,  $x \neq 0$ . Consequently, the set  $\{\Lambda_{\tilde{\Phi}}; \tilde{\Phi} \text{ is plh and } \Phi - \tilde{\Phi} > 0\}$  is a connected component of the set of all  $\mathbf{C}$ -Lagrangian spaces that are transversal to  $\Lambda_{\Phi}$ . It contains  $\Lambda_{\Phi_{\text{plh}}}$  which is  $\Lambda_{\Phi}$ -positive, so by Corollary A.7 it is equal to the set of all  $\mathbf{C}$ -Lagrangian spaces, which are  $\Lambda_{\Phi}$ -positive.  $\square$

## B Quadratic Fourier integral operators in the complex domain

The Fourier integral operators encountered in this appendix arise when quantizing complex linear canonical transformations. The following discussion can therefore be viewed as a linear version of the theory presented in Chapters 3 and 4 in [26]. See also Chapter 3 of [20].

We start with the formal theory. A Fourier integral operator is an operator of the form

$$Au(x) = \iint e^{i\phi(x,y,\theta)} au(y) dy d\theta, \quad (\text{B.7})$$

where  $a \in \mathbf{C}$  and  $\phi$  is a holomorphic quadratic form on  $\mathbf{C}_{x,y,\theta}^{n+n+N}$ . We assume that  $\phi$  is a nondegenerate phase function in the sense of [17], so that

$$d\frac{\partial\phi}{\partial\theta_1}, \dots, d\frac{\partial\phi}{\partial\theta_N} \text{ are linearly independent,} \quad (\text{B.8})$$

or equivalently that

$$\phi''_{\theta,(x,y,\theta)} \text{ is of rank } N. \quad (\text{B.9})$$

We may assume without loss of generality that

$$\phi''_{\theta,\theta} = 0. \quad (\text{B.10})$$

Indeed, if  $\phi''_{\theta,\theta} \neq 0$ , we may assume after a linear change of coordinates in  $\theta$ , that  $\theta = (\theta', \theta'') \in \mathbf{C}^{N-d} \times \mathbf{C}^d$ ,  $\phi''_{\theta',\theta} = 0$  and that  $\det \phi''_{\theta'',\theta''} \neq 0$ . Then  $\theta'' \mapsto \phi(x, y, \theta)$  has a unique critical point  $\theta''_c(x, y)$  which is nondegenerate and integrating out the  $\theta''$ -variables in (B.7), we get formally

$$Au(x) = \iint e^{i\psi(x,y,\theta')} bu(y) dy d\theta', \quad (\text{B.11})$$

where  $b$  is a constant non-vanishing multiple of  $a$  and  $\psi(x, y, \theta') = \phi(x, y, \theta', \theta''_c)$  is a nondegenerate phase function with  $\psi''_{\theta',\theta'} = 0$ .

Assuming (B.10) until further notice, we see that (B.9) becomes

$$\text{rank}(\phi''_{\theta,(x,y)}) = N, \quad (\text{B.12})$$

and in particular,  $N \leq 2n$ .

Let

$$C_\phi = \{(x, y, \theta) \in \mathbf{C}^{2n+N}; \phi'_\theta(x, y, \theta) = 0\} \quad (\text{B.13})$$

This is a subspace of dimension  $2n$  and we consider the corresponding canonical relation

$$\kappa : (y, \eta) \mapsto (x, \xi), \quad (\text{B.14})$$

defined by its graph

$$\text{graph}(\kappa) = \{(x, \xi; y, \eta) = (x, \phi'_x(x, y, \theta); y, -\phi'_y(x, y, \theta)); (x, y, \theta) \in C_\phi\}. \quad (\text{B.15})$$

It is easy to check that  $\dim(\text{graph}(\kappa)) = 2n$  and that

$$(\sigma_{x,\xi} - \sigma_{y,\eta})|_{\text{graph}(\kappa)} = 0,$$

where  $\sigma_{x,\xi} - \sigma_{y,\eta} = \sum_1^n d\xi_j \wedge dx_j - \sum_1^n d\eta_j \wedge dy_j$ .

Assume that

$$\kappa \text{ is a canonical transformation,} \quad (\text{B.16})$$

which means that the maps  $\text{graph}(\kappa) \ni (x, \xi; y, \eta) \mapsto (x, \xi) \in \mathbf{C}^{2n}$  and  $\text{graph}(\kappa) \ni (x, \xi; y, \eta) \mapsto (y, \eta) \in \mathbf{C}^{2n}$  are bijective. Actually, the bijectivity of one of the maps implies that of the other. In fact, the bijectivity of the first map is equivalent to that of  $C_\phi \ni (x, y, \theta) \mapsto (x, \phi'_x(x, y, \theta)) \in \mathbf{C}^{2n}$  which is equivalent to that of

$$\mathbf{C}^{2n+N} \ni (x, y, \theta) \mapsto (x, \phi'_x, \phi'_\theta) = (x, \phi''_{x,x}x + \phi''_{x,y}y + \phi''_{x,\theta}\theta, \phi''_{\theta,x}x + \phi''_{\theta,y}y) \in \mathbf{C}^{2n+N}.$$

This is equivalent to the bijectivity of

$$\begin{pmatrix} \phi''_{x,y} & \phi''_{x,\theta} \\ \phi''_{\theta,y} & 0 \end{pmatrix}.$$

and implies that  $\phi''_{\theta,y}$  and  $\phi''_{x,\theta}$  are of maximal rank  $N$  and in particular that  $N \leq n$ .

We can write

$$\phi(x, y, \theta) = g(x, y) + \sum_{j=1}^N f_j(x, y)\theta_j, \quad (\text{B.17})$$

where  $g$  is quadratic and  $f_j$  are linear. Further,  $d_x f_1, \dots, d_x f_N$  are linearly independent and similarly for the  $y$ -differentials. The subspace  $C_\phi$  is given by the equations  $f_j(x, y) = 0$ ,  $j = 1, \dots, N$  in  $\mathbf{C}^{2n+N}$  and the same equations in the  $x, y$  space define the projection of  $C_\phi$ , or equivalently, the projection  $\pi_{x,y}(\text{graph}(\kappa))$ , so  $N$  is uniquely determined by  $\kappa$ . Let

$$\tilde{\phi}(x, y, \tilde{\theta}) = \tilde{g}(x, y) + \sum_{j=1}^N \tilde{f}_j(x, y)\tilde{\theta}_j \quad (\text{B.18})$$

be a second nondegenerate phase function that generates the same canonical transformation  $\kappa$ . Then  $f_1 = \dots = f_N = 0$  and  $\tilde{f}_1 = \dots = \tilde{f}_N = 0$  define the same subspace of  $\mathbf{C}^{2n}$  and after a linear change of the  $\tilde{\theta}$  coordinates, we may assume that

$$\tilde{\phi} = \tilde{\phi}(x, y, \theta) = \tilde{g}(x, y) + \sum_{j=1}^N f_j(x, y)\theta_j.$$

The fact that the two phases give rise to the same canonical transformation now means that there are linear forms  $k_j(x, y)$ ,  $1 \leq j \leq N$  such that

$$\tilde{g}(x, y) = g(x, y) + \sum_{j=1}^N f_j(x, y)k_j(x, y). \quad (\text{B.19})$$

Then

$$\tilde{\phi}(x, y, \theta) = \phi(x, y, \theta + k(x, y)), \quad (\text{B.20})$$

which will imply that  $\tilde{\phi}$  gives rise to the same Fourier integral operators, once we have explained how to choose the contour of integration in (B.7).

Let  $\Phi_0$  and  $\Phi_1$  be real quadratic forms on (different copies of)  $\mathbf{C}^n$ .

**Proposition B.1** *Let  $\phi$  be a nondegenerate phase function as above, with the associated canonical transformation  $\kappa$ . The following statements are equivalent:*

$$1) \quad \kappa(\Lambda_{\Phi_0}) = \Lambda_{\Phi_1}.$$

2) *The quadratic form*

$$(y, \theta) \mapsto \Phi_0(y) - \text{Im } \phi(0, y, \theta) \quad (\text{B.21})$$

*is nondegenerate so that*

$$(y, \theta) \mapsto \Phi_0(y) - \text{Im } \phi(x, y, \theta) \quad (\text{B.22})$$

*has a unique critical point  $(y(x), \theta(x))$  for every  $x \in \mathbf{C}^n$ . Moreover,*

$$\Phi_1(x) = \text{vc}_{y, \theta}(\Phi_0(y) - \text{Im } \phi(x, y, \theta)).$$

*Proof:* We first assume 2). Then at the critical point  $(y(x), \theta(x))$ , we have

$$\begin{aligned} \frac{2}{i} \frac{\partial \Phi_0}{\partial y} &= -\frac{2}{i} \frac{\partial}{\partial y} (-\text{Im } \phi(x, y, \theta)) = -\frac{\partial}{\partial y} \phi(x, y, \theta), \\ \frac{\partial}{\partial \theta} \phi(x, y, \theta) &= 0, \\ \frac{2}{i} \frac{\partial \Phi_1}{\partial x} &= \frac{2}{i} \frac{\partial}{\partial x} (-\text{Im } \phi)(x, y, \theta) = \frac{\partial \phi}{\partial x}(x, y, \theta), \end{aligned}$$

which means that 1) holds.

Now assume 1). Then for every  $x \in \mathbf{C}^n$  there exists a unique  $y \in \mathbf{C}^n$  such that  $x \in \pi_x(\kappa(y, \frac{2}{i} \frac{\partial \Phi_0}{\partial y}(x)))$ . Equivalently for every  $x \in \mathbf{C}^n$ , (B.22) has a unique critical point  $(y(x), \theta(x))$ . Since we are dealing with second order polynomials this critical point is nondegenerate. If  $\tilde{\Phi}_1(x)$  is the critical value, we also see that

$$\frac{2}{i} \frac{\partial \Phi_1}{\partial x} = \frac{2}{i} \frac{\partial \tilde{\Phi}_1}{\partial x}$$

and since  $\Phi_1$  and  $\tilde{\Phi}_1$  are quadratic forms, we conclude that  $\tilde{\Phi}_1 = \Phi_1$ .  $\square$

If  $\kappa(\Lambda_{\Phi_0}) = \Lambda_{\Phi_1}$  and  $\Phi_0$  is pluriharmonic (so that  $\Lambda_{\Phi_0}$  is a  $\mathbf{C}$ -Lagrangian space), then  $\Phi_1$  is also pluriharmonic and the nondegenerate quadratic form (B.21) has the signature  $(n + N, n + N)$ . In general, when  $\Phi_0(y)$  and hence  $\Phi_0(y) - \text{Im } \phi(0, y, \theta)$  is plurisubharmonic, then (B.21) has at most  $n + N$  negative eigenvalues, and in order to find a suitable integration contour, when realizing  $A$ , we wish this maximal number of negative eigenvalues to be attained. The following result gives a positive answer in the case that we are interested in.

**Proposition B.2** *Assume that  $\kappa(\Lambda_{\Phi_0}) = \Lambda_{\Phi_1}$ , where  $\partial_{\bar{z}}\partial_z\Phi_0$  and  $\partial_{\bar{z}}\partial_z\Phi_1$  are positive definite. Then the signature of the quadratic form (B.21) is equal to  $(n + N, n + N)$ .*

*Proof:* Let  $p$  be a positive definite quadratic form on  $\Lambda_{\Phi_0}$  and let  $p$  also denote the holomorphic extension to  $\mathbf{C}^{2n}$  (which is a well defined holomorphic quadratic form since  $\Lambda_{\Phi_0}$  is maximally totally real). Then the spectrum of the fundamental matrix  $F_p$  is given by the eigenvalues  $\pm i\mu_1, \dots, \pm i\mu_n$ , where  $\mu_j > 0$ . Let  $\Lambda_{0,+ \infty}$  denote the spectral subspace of  $\mathbf{C}^{2n}$  corresponding to  $i\mu_1, \dots, i\mu_n$ . Then we know from [25] that  $\Lambda_{0,+ \infty}$  is positive with respect to  $\Lambda_{\Phi_0}$ , so that, according to Proposition A.9,  $\Lambda_{0,+ \infty}$  is of the form  $\xi = \frac{2}{i} \frac{\partial \Phi_{0,+ \infty}}{\partial x}(x)$ , where  $\Phi_{0,+ \infty}$  is a pluriharmonic quadratic form such that  $\Phi_0(x) - \Phi_{0,+ \infty}(x) \asymp |x|^2$ .

The spectral subspace  $\Lambda_{0,+ \infty}$  can be viewed as the unstable manifold for the  $H_{-ip}$ -flow and if we put  $\Lambda_{0,t} = \exp(-itH_p)(\Lambda_{\Phi_0})$ , for  $t \in \mathbf{R}$ , then  $\Lambda_{0,t}$  is I-Lagrangian and R-symplectic, and  $\Lambda_{0,t}$  converges to  $\Lambda_{0,+ \infty}$  exponentially fast. Furthermore, at least for small  $t > 0$ , we have  $\Lambda_{0,t} = \Lambda_{\Phi_{0,t}}$ , where  $\Phi_{0,t}(x)$  is the real quadratic form obtained by solving the eikonal equation

$$\frac{\partial}{\partial t} \Phi_{0,t}(x) + \text{Re } p \left( x, \frac{2}{i} \frac{\partial}{\partial x} \Phi_{0,t}(x) \right) = 0. \quad (\text{B.23})$$

See for instance, [27], as well as Remark 11.7 in [13], for more details. Notice here that since  $p$  is constant along the  $H_{-ip}$  trajectories, we have  $\text{Re } p = p$  in (B.23) and  $p \left( x, \frac{2}{i} \frac{\partial}{\partial x} \Phi_{0,t}(x) \right) \asymp |x|^2$ , non-uniformly with respect to  $t$ .

A priori it is not clear that we can solve (B.23) for all  $t \geq 0$ . However, as long as the solution exists, we see that  $t \mapsto \Phi_{0,t}$  is a decreasing family of strictly plurisubharmonic quadratic forms, so that  $\Phi_{0,t}(x) \leq \Phi_0(x)$ . For such a  $t$ , we write  $\psi := \Phi_{0,t} = \psi_{\text{herm}} + \psi_{\text{plh}}$ , where  $\psi_{\text{herm}}(x) = \frac{1}{2}(\psi(x) + \psi(ix))$  is the Hermitian



part and  $\psi_{\text{plh}}(x) = \frac{1}{2}(\psi(x) - \psi(ix))$  is the pluriharmonic part. Similarly, we write  $\phi := \Phi_0(x) = \phi_{\text{herm}} + \phi_{\text{plh}}$ . From  $\psi \leq \phi$  we get  $0 \leq \psi_{\text{herm}} \leq \frac{1}{2}(\phi(x) + \phi(ix)) = \phi_{\text{herm}}$ . Also,

$$\begin{aligned}\psi_{\text{plh}}(x) &\leq (\phi_{\text{herm}}(x) - \psi_{\text{herm}}(x)) + \phi_{\text{plh}}(x) \leq \phi(x), \\ -\psi_{\text{plh}}(x) &= \psi_{\text{plh}}(ix) \leq \phi(ix),\end{aligned}$$

so that

$$-\phi(ix) \leq \psi_{\text{plh}}(x) \leq \phi(x).$$

This means that we have  $t$ -independent upper and lower bounds on  $\Phi_{0,t}$ , which prevent an explosion in the eikonal equation (B.23). Consequently,  $\Lambda_{0,t} = \Lambda_{\Phi_{0,t}}$  for all  $0 \leq t < \infty$ , and the exponential convergence of  $\Lambda_{0,t}$  to  $\Lambda_{0,+\infty}$  when  $t \rightarrow +\infty$  shows that

$$\Phi_{0,t} \rightarrow \Phi_{0,+\infty} \text{ exponentially fast, when } t \rightarrow +\infty.$$

Let us define  $q$  on  $\mathbf{C}^{2n}$  by  $q \circ \kappa = p$ . Replacing  $(\Phi_0, p)$  in the above discussion by  $(\Phi_1, q)$ , we get a decreasing family of strictly plurisubharmonic quadratic forms  $\Phi_{1,t}$  for  $0 \leq t < +\infty$ , converging to the pluriharmonic form  $\Phi_{1,+\infty}$  when  $t \rightarrow +\infty$ , with the property that

$$\kappa(\Lambda_{\Phi_{0,t}}) = \Lambda_{\Phi_{1,t}}, \quad 0 \leq t \leq +\infty.$$

Proposition B.1 now shows that

$$(y, \theta) \mapsto \Phi_{0,t}(y) - \text{Im } \phi(0, y, \theta)$$

is a nondegenerate quadratic form for  $0 \leq t \leq +\infty$ , necessarily with  $t$ -independent signature. When  $t = +\infty$  the signature is equal to  $(n + N, n + N)$  so this is also the case when  $t = 0$ , which gives the proposition.  $\square$

Now let us recall the representation (B.17), where  $\phi''_{\theta,y}$  and  $\phi''_{\theta,x}$  are of maximal rank  $N$ . After separate linear changes of the  $x$  and  $y$  coordinates, we may assume, in order to simplify the notation only, that

$$\phi(x, y, \theta) = g(x, y) + \sum_{j=n-N+1}^N (x_j - y_j)\theta_j = g(x, y) + (x'' - y'') \cdot \theta, \quad (\text{B.24})$$

writing  $x = (x', x'') \in \mathbf{C}^{n-N} \times \mathbf{C}^N$  and similarly for  $y$ . We see directly that

$$-\text{Im } \phi(0, 0, y'', \theta) + \Phi_0(0, y'') = -\text{Im } g(0, 0, y'') + \Phi_0(0, y'') + \text{Im } (y'' \cdot \theta)$$

is a nondegenerate quadratic form of signature  $(N, N)$ . More generally, the second order polynomial

$$(y'', \theta) \mapsto -\text{Im } \phi(x, y, \theta) + \Phi_0(y) = -\text{Im } g(x, y) + \Phi_0(y) - \text{Im } ((x'' - y'') \cdot \theta) \quad (\text{B.25})$$

has a unique critical point of signature  $(2N, 2N)$ . The critical point  $(y'_c(x, y'), \theta_c(x, y'))$  is given by  $y'_c = x''$ ,  $\theta_c = \frac{2}{i} \frac{\partial \Phi_0}{\partial y''}(y', x'') + \frac{\partial g}{\partial y''}(x, y', x'')$  and the critical value is

$$-\text{Im } \phi(x, y', x'', \theta_c) + \Phi_0(y', x'') = -\text{Im } g(x, y', x'') + \Phi_0(y', x''). \quad (\text{B.26})$$

On the other hand, by Proposition B.2, we know that

$$(y, \theta) \mapsto -\text{Im } \phi(x, y, \theta) + \Phi_0(y) \quad (\text{B.27})$$

has a unique critical point which is of signature  $(n + N, n + N)$ . It follows that

$$y' \mapsto -\text{Im } g(x, y', x'') + \Phi_0(y', x'') \quad (\text{B.28})$$

has a unique critical point which is of signature  $(n - N, n - N)$ , and the corresponding critical value is equal to that of (B.27), namely  $\Phi_1(x)$ .

Let  $\gamma \subset \mathbf{C}_{y'}^{n-N}$  be a real-linear subspace of dimension  $n - N$  such that

$$-\text{Im } g(0, y', 0) + \Phi_0(y', 0) \asymp -|y'|^2, \quad y' \in \gamma. \quad (\text{B.29})$$

Then

$$-\text{Im } g(x, y', x'') + \Phi_0(y', x'') - \Phi_1(x) \asymp -|y' - y'_c(x)|^2, \quad y' \in y_c(x) + \gamma. \quad (\text{B.30})$$

Let us consider the contour  $\Gamma(x) \subset \mathbf{C}_{y, \theta}^{n+N}$  of real dimension  $n + N$ , given by

$$y' \in y'_c(x) + \gamma, \quad y'' \in \mathbf{C}^N, \quad \theta = \theta_c(x, y') + iC \overline{(x'' - y'')},$$

parametrized by  $(y', y'') \in \gamma \times \mathbf{C}^N$ . Along  $\Gamma(x)$  we have

$$-\text{Im } \phi(x, y, \theta) + \Phi_0(y) - \Phi_1(x) \asymp -(|y' - y'_c(x)|^2 + |y'' - x''|^2), \quad (\text{B.31})$$

provided that we choose  $C > 0$  large enough.

Let  $u \in H_{\Phi_0}(\mathbf{C}^n)$ , and let us realize  $Au = A_\Gamma u$  in (B.7) by integrating over  $\Gamma(x)$ . We get

$$\begin{aligned} e^{-\Phi_1(x)} A_\Gamma u(x) &= \int_{y'' \in \mathbf{C}^N} \int_{y' \in y'_c(x) + \gamma} e^{i\psi_\Gamma(x, y)} \tilde{a}(u(y) e^{-\Phi_0(y)}) dy' L(dy'') \\ &= e^{-\Phi_1} A_\Gamma e^{\Phi_0}(u e^{-\Phi_0}), \end{aligned} \quad (\text{B.32})$$

where  $-\text{Im } \psi_\Gamma(x, y)$  is equal to the right hand side in (B.31). From (B.31) we see that the integral in (B.32) converges for each  $x \in \mathbf{C}^n$ , and by a standard application of Stokes' formula, we also know that if  $\tilde{\Gamma}$  is a second contour with the same properties as  $\Gamma$ , then  $A_\Gamma u(x) = A_{\tilde{\Gamma}} u(x)$ .

We now wish to estimate the  $\mathcal{L}(L^2, L^2)$ -norm of the effective operator  $e^{-\Phi_1} A_\Gamma e^{\Phi_0}$  in (B.32). We know that the map

$$x \mapsto y_c(x) = (y'_c(x), x'') \quad (\text{B.33})$$

is bijective. Replacing  $x$  by the new  $x$  coordinates  $(y'_c(x), x'')$ , we may reduce ourselves to the case when  $y'_c(x) = x'$ . After a simultaneous rotation in the  $x'$  and  $y'$  variables we may further assume that  $\gamma = \mathbf{R}^{n-N}$ . Then from (B.31), (B.32), we get

$$e^{-\Phi_1(x)} A_\Gamma u(x) = \iint_{\substack{t' \in \mathbf{R}^{n-N} \\ y'' \in \mathbf{C}^N}} \mathcal{O}(1) e^{-\frac{1}{C}(|\text{Re } x' - t'|^2 + |x'' - y''|^2)} e^{-\Phi_0(t' + i\text{Im } x', y'')} u(t' + i\text{Im } x', y'') dt' L(dy''). \quad (\text{B.34})$$

By Schur's lemma,

$$\|e^{-\Phi_1} A_\Gamma u\|_{L^2(E_{t'})} \leq \mathcal{O}(1) \|e^{-\Phi_0} u\|_{L^2(E_{t'})},$$

uniformly for  $t' \in \mathbf{R}^{n-N}$ , where  $E_{t'} = \{x \in \mathbf{C}^n; \text{Im } x' = t'\}$ . Integrating this with respect to  $t'$ , we get

$$\|e^{-\Phi_1} A_\Gamma u\|_{L^2(\mathbf{C}^n)} \leq \mathcal{O}(1) \|e^{-\Phi_0} u\|_{L^2(\mathbf{C}^n)}.$$

We have proved the following result.

**Proposition B.3** *Realizing the operator  $A$  by means of the contour  $\Gamma$  above, we get a bounded operator  $A = A_\Gamma : H_{\Phi_0} \rightarrow H_{\Phi_1}$ . We get the same operator if we replace  $\Gamma$  by any other contour with the same properties.*

The next result can be proved by the general methods of [26].

**Proposition B.4** *Let  $(B, \tilde{\kappa})$  have same general properties as  $(A, \kappa)$  and assume that  $\tilde{\kappa}(\Lambda_{\Phi_1}) = \Lambda_{\Phi_2}$ , where  $\Phi_2$  is a strictly plurisubharmonic quadratic form. Let  $\tilde{\Gamma}$  be a contour allowing to realize  $B = B_{\tilde{\Gamma}} : H_{\Phi_1} \rightarrow H_{\Phi_2}$ . Then there exists a Fourier integral operator  $C$  as above with associated canonical transformation  $\tilde{\kappa} \circ \kappa$  and a contour  $\hat{\Gamma}$  such that  $C_{\hat{\Gamma}} = B_{\tilde{\Gamma}} \circ A_\Gamma$ .*

*Proof:* We shall only recall the general ideas of the proof. First of all, the elimination of superfluous  $\theta$  coordinates above was only in order to simplify the presentation, and it is very easy to realize Fourier integral operators with more than the minimal number of such fiber variables. Formally  $B \circ A$  then becomes such a Fourier integral operator with  $\tilde{\kappa} \circ \kappa$  as the associated canonical transformation. When composing  $B_{\tilde{\Gamma}} \circ A_{\Gamma}$  we automatically get a good contour, and as we have pointed out, this contour can be deformed into any other good contour for  $B \circ A$ . Superfluous  $\theta$  variables can then be eliminated by the exact version of stationary phase.  $\square$

A special case is when  $\kappa = \text{id}$ . We then get a multiple of the identity operator. If  $\tilde{\kappa}$  in the last proposition is equal to  $\kappa^{-1}$ , then we conclude that  $B \circ A$  is a multiple of the identity, and assuming that the amplitude  $a$  in (B.7) is  $\neq 0$ , we can choose the amplitude  $b$  in a corresponding representation of  $B$  in such a way that  $B \circ A = I$ . In particular, we see that  $A_{\Gamma} : H_{\Phi_0} \rightarrow H_{\Phi_1}$  in Proposition B.3 has a bounded (two-sided) inverse.

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